Proposition 3.2. *Let the i^{th} contact be loaded along a linear loading profile with slope c_i. If the loading profile satisfies the inequality*

$$|c_i| < 2\sqrt{\frac{3(2-\nu)}{(1-\nu)}}, \tag{8}$$

the symmetrized stiffness matrix induced by the i^{th} contact, $\left(\tilde{K}_i(q_0)\right)_s$, is negative semi-definite.

Proof: Denote by A_i the 2×2 matrix at the core of $\tilde{K}_i(q_0)$ in (7):

$$A_i = \begin{bmatrix} \frac{4}{3(2-\nu)} & -\frac{2c_i}{3(2-\nu)} \\ 0 & \frac{1}{1-\nu} \end{bmatrix}.$$

Then $(\tilde{K}_i)_s = -G_i^T R_i (A_i)_s R_i^T G_i$, where we omitted the positive scalars preceding $\tilde{K}_i$. The positive definiteness of $(A_i)_s$ implies the negative semi-definiteness of $(\tilde{K}_i)_s$. The 2×2 matrix $(A_i)_s$ is positive definite when its two eigenvalues, λ_1 and λ_2, are strictly positive. First consider the trace of $(A_i)_s$, $\mathsf{tr}(A_i)_s = \lambda_1 + \lambda_2$. Inspection of the diagonal elements in $(A_i)_s$ reveals that $\mathsf{tr}(A_i)_s > 0$ when $\nu < 1$. Since Poisson's ratio satisfies $0 \leq \nu \leq 0.5$ for all practical materials, $\mathsf{tr}(A_i)_s > 0$. Next consider the determinant of $(A_i)_s$, $\det(A_i)_s = \lambda_1 \lambda_2$. Since $\lambda_1 + \lambda_2 > 0$, a positive determinant would imply positive definiteness of $(A_i)_s$. The determinant of $(A_i)_s$ is:

$$\det(A_i)_s = \frac{4}{3(1-\nu)(2-\nu)} - \frac{c_i^2}{9(2-\nu)^2}.$$

The inequality $\det(A_i)_s > 0$ becomes $4/(1-\nu) > c_i^2/3(2-\nu)$. A square root of both sides gives the condition for the positive definiteness of $(A_i)_s$, which implies the negative semi-definiteness of $(\tilde{K}_i)_s$. □

Example: Consider the two constraints imposed on the slope of the linear loading path. The term $(2-\nu)/(1-\nu)$ which appears in (8) varies in the interval $[2,3]$ (since $0 \leq \nu \leq 0.5$). Hence $(\tilde{K}_i)_s$ is negative semi-definite when $|c_i| < 2\sqrt{6}$. The corresponding slope angle, denoted β in Figure 3, must satisfy $|\beta| < 78.5°$. The slope of the linear loading path must also satisfy the friction cone constraint specified in (3), $|c_i| < \mu(2-\nu)/2(1-\nu)$. Since $(2-\nu)/(1-\nu)$ varies in the interval $[2,3]$, the friction cone constraint is satisfied when $|c_i| < \mu$. Since the linear loading path must always satisfy the friction cone constrain, the negative semi-definiteness of $(\tilde{K}_i)_s$ is automatic as long as $\mu \leq 2\sqrt{6}$, which clearly holds true in most practical situations. For instance, when $\mu = 1$ the friction cone constraint requires that $|c_i| < 1$. The corresponding slope angle, denoted α in Figure 3, must satisfy $|\alpha| < 45°$. The matrix $(\tilde{K}_i)_s$ is thus negative semi-definite in typical linearly loaded grasps.